The collision of two ionized streams

By F. D. KAHN
Department of Astronomy, Physical Laboratories, University of Manchester

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SUMMARY

It is shown that the interpenetration of two ionized streams is arrested, as a rule, not because of individual collisions between particles belonging to opposite streams, but because the whole system of charged particles is unstable. The smallest wavelength of an unstable oscillation is $\lambda_{\min}$, where

$$\lambda_{\min} = \sqrt{\left(\frac{\pi m_0 U^2}{2N\epsilon^2}\right)}\left(1-\frac{U^2}{c^2}\right)^{-3/4}.$$

Here $\pm U$ are the velocities of the undisturbed streams, and N is the density of electrons in each.

A further calculation for the non-relativistic case deals with the amplification of the plasma oscillations present in two colliding streams. It is shown that these grow rapidly and that $\tau_{\text{crit}}\sqrt{(m_0 U^2/\pi N\epsilon^2)}$ is the distance of interpenetration achieved before the counterstreaming of the electrons is brought to a halt. The value of τ_{crit} depends only insensitively on the ratio of the internal plasma energy densities T_{pl} to the kinetic energy densities T_{kin} in the streams. For example, $\tau_{\text{crit}} = 9{\cdot}0$ when $T_{\text{pl}}:T_{\text{kin}} = 1:10$, and $\tau_{\text{crit}} = 19{\cdot}0$ when $T_{\text{pl}}:T_{\text{kin}} = 1:10^5$.

1. INTRODUCTION

The interpenetration of two fully ionized streams of gas is considered in this paper. It is shown that, as a rule, the counterstreaming is stopped because of a collective instability among the electrons present, and that close encounters between charged particles belonging to the two streams are relatively unimportant in destroying the systematic motion.

Advance information of some of the results in this paper was given by the author in an earlier paper (Kahn 1955) where it was suggested that this instability effect leads to the conversion of the kinetic energy of the two streams into the energy of irregular plasma oscillations. A particular example of this might be found in the collision of two galaxies—as in the radio source Cygnus A. One would therefore expect random electric fields to be present in the plasma left after a collision. A fast charged particle passing through such a medium would then be expected to radiate electromagnetic waves.

The instability effect may also be important in the laboratory. It may, for instance, explain the smallness of the distance to which two counterstreaming masses of plasma can interpenetrate. That the distance is small is shown, for instance, by Bostick's (1956) experiments.

In the following two sections we derive the condition that an oscillation of a particular wavelength may be amplified. The treatment given is non-relativistic in § 2, and relativistic in § 3. In § 4 we consider the amplification of the oscillations which are present in the two streams before they have collided, and find a closer estimate of the time that elapses before the counterstreaming is stopped.

Some calculations concerning this effect have been given by Lampert (1956), who uses the dispersion relations for a plasma.

2. A NON-RELATIVISTIC TREATMENT

Consider two interpenetrating streams, each of infinite extent, and each consisting of N protons and N electrons per cm^3. Let stream 1 have a velocity U in the direction of increasing x, and stream 2 a velocity U in the direction of decreasing x. Let ϵ be the charge on the electron and m_0 its rest mass. Let the temperature T in each stream be such that $kT \ll m_0 U^2$. The thermal motion among the particles may then be neglected.

An electron passing within a distance r of a charged particle from the other stream experiences a change ϵ^2/r in potential energy. Its kinetic energy relative to this particle is $\frac{1}{2}m_0(2U)^2 = 2m_0 U^2$. An appreciable deflection in the direction of the electron's motion occurs if

$$\epsilon^2/r = O(2m_0 U^2),$$

that is,

$$r = O(\epsilon^2/2m_0 U^2). \tag{1}$$

The effective cross-sectional area for such a close collision is thus of the order of $S = \pi\epsilon^4/4m_0^2 U^4$. These encounters would eventually destroy the systematic motion. The distance of interpenetration will at most be of the order of

$$\frac{1}{2NS} = \frac{2m_0^2 U^4}{\pi N \epsilon^4}. \tag{2}$$

But there is a collective instability among the electrons which will stop the counterstreaming very much sooner. Assume, for the moment, that the protons, because of their much greater mass, continue their uniform motion, and provide a uniform background of positive charge. Let the densities in the electron streams be $N(1+s_1)$ and $N(1+s_2)$, respectively, and let the velocities be $U+u_1$ and $-U+u_2$; s_1, s_2, u_1 and u_2 are here regarded as functions of position and time. We shall confine ourselves to one-dimensional disturbances, so that the position enters through the x-coordinate only. The resulting electrostatic field will be parallel to Ox; let E be its magnitude.

Poisson's equation then gives

$$\frac{\partial E}{\partial x} = 4\pi N\epsilon(s_1+s_2), \tag{3}$$

and the linearized equations of motion for the two electron streams are

$$\frac{\partial u_1}{\partial t} + U\frac{\partial u_1}{\partial x} = \frac{\epsilon}{m}E, \tag{4}$$

$$\frac{\partial u_2}{\partial t} - U\frac{\partial u_2}{\partial x} = \frac{\epsilon}{m}E. \tag{5}$$

The equations of continuity for the two electron streams are

$$\left[\frac{\partial}{\partial t} + (U+u_1)\frac{\partial}{\partial x}\right]N(1+s_1) + N(1+s_1)\frac{\partial u_1}{\partial x} = 0,$$

$$\left[\frac{\partial}{\partial t} - (U-u_2)\frac{\partial}{\partial x}\right]N(1+s_2) + N(1+s_2)\frac{\partial u_2}{\partial x} = 0,$$

or, in linearized form,

$$\frac{\partial s_1}{\partial t} + U\frac{\partial s_1}{\partial x} + \frac{\partial u_1}{\partial x} = 0, \tag{6}$$

$$\frac{\partial s_2}{\partial t} - U\frac{\partial s_2}{\partial x} + \frac{\partial u_2}{\partial x} = 0. \tag{7}$$

The combination of (3) and (4) leads to

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\frac{\partial u_1}{\partial x} = \Omega^2(s_1+s_2), \tag{8}$$

where $\Omega^2 = 4\pi N\epsilon^2/m_0$. With the aid of (6),

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2 s_1 = -\Omega^2(s_1+s_2). \tag{9}$$

Similarly,

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 s_2 = -\Omega^2(s_1+s_2). \tag{10}$$

Now look for solutions of (9) and (10) in the form

$$s_1 = a_1 e^{i(kx-\omega t)}, \qquad s_2 = a_2 e^{i(kx-\omega t)}. \tag{11}$$

If a solution can be found, with a given real k, for which $\omega = \mu + i\sigma$, where μ and σ are real, and $\sigma > 0$, then the counterstreaming is unstable, for the amplitude of the corresponding oscillation can grow indefinitely. The maximum value $k_{\max}$ for which this is possible gives the minimum value $\lambda_{\min}$ of the wavelengths at which there is instability, through the relation $\lambda_{\min} = 2\pi/k_{\max}$. If such counterstreaming were ever to occur it would be stopped within a distance of the order of $\lambda_{\min}$.

The evaluation of $k_{\max}$ is straightforward. Substitution from (11) into (9) leads to

$$(\omega+kU)^2 a_1 = \Omega^2(a_1+a_2),$$

or

$$[(\omega+kU)^2 - \Omega^2]a_1 - \Omega^2 a_2 = 0. \tag{12}$$

Similarly, from (10),

$$-\Omega^2 a_1 + [(\omega-kU)^2 - \Omega^2]a_2 = 0. \tag{13}$$

Elimination of a_1 and a_2 from (12) and (13) gives

$$[(\omega+kU)^2 - \Omega^2][(\omega-kU)^2 - \Omega^2] - \Omega^4 = 0,$$

or

$$\omega^4 - 2\omega^2(k^2U^2+\Omega^2) + k^2U^2(k^2U^2 - 2\Omega^2) = 0. \tag{14}$$

Since $$(k^2U^2+\Omega^2)^2 \geqslant k^2U^2(k^2U^2-2\Omega^2)$$
for all real values of k, U and Ω, it follows that (14) always has real roots for ω^2. However, if $k^2U^2-2\Omega^2$ is negative, one possible value of ω^2 is negative, say $-\sigma^2$, and this leads to

$$\omega = \pm i\sigma,$$

with instability as a consequence.

The critical wave-number is given by $k^2U^2-2\Omega^2 = 0$, so that

$$k_{\max} = \sqrt{2}\Omega/U.$$

The corresponding critical wavelength is

$$\lambda_{\min} = 2\pi/k_{\max} = \sqrt{2}\pi U/\Omega.$$

Any disturbance with a wavelength exceeding $\lambda_{\min}$ will be amplified. The counterstreaming thus becomes unstable within a distance of the order of $\lambda_{\min}$. In terms of N, ϵ, m_0 and U,

$$\lambda_{\min} = \sqrt{\left(\frac{\pi m_0 U^2}{2N\epsilon^2}\right)}. \tag{15}$$

This is much shorter than the limit set to the counterstreaming by close collisions if

$$\frac{2m_0^2 U^4}{\pi N\epsilon^4} \ll \sqrt{\left(\frac{\pi m_0 U^2}{2N\epsilon^2}\right)},$$

that is, if $$N \ll \frac{8m_0^3 U^6}{\pi^3\epsilon^6}. \tag{16}$$

To give a numerical value, when $U = 10^8$ cm/s the condition states that N need only be much smaller than 2×10^{22} particles per cm^3. This critical density is very large. The instability effect is therefore usually the one that arrests the counterstreaming of the electrons.

It is harder to say exactly what happens to the protons. They have a larger value of $\lambda_{\min}$, but in their case the greater effectiveness of the instability, as opposed to that of close collisions, is even more marked (as is shown by substitution of a larger value for m_0 in (15)). However the presence of the oscillating electron gas cannot be left out of account in any calculation, and the theory is rather harder. It seems nonetheless clear that the protons also contribute their kinetic energy to the collective oscillation, rather than to thermal motion.

3. The extension to the relativistic case

Let $U+u_1$ be the velocity of an electron in the first stream. Its momentum is

$$p_1 = \frac{m_0(U+u_1)}{\{1-(U+u_1)^2/c^2\}^{1/2}} = \frac{m_0 U}{(1-U^2/c^2)^{1/2}} + \frac{m_0 u_1}{(1-U^2/c^2)^{1/2}}\left(1+\frac{U^2/c^2}{1-U^2/c^2}\right)+\dots$$
$$= \frac{m_0 U}{(1-U^2/c^2)^{1/2}} + \frac{m_0 u_1}{(1-U^2/c^2)^{3/2}}+\dots$$

to the first order. The linearized equation of motion is

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)p_1 = \epsilon E,$$

or

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)u_1 = \epsilon E/M,$$

where $M = m_0(1-U^2/c^2)^{-3/2}$. There is a similar change in the equation for the second stream. Thus, the only alteration is that the rest mass m_0 is replaced by a virtual mass $M = m_0(1-U^2/c^2)^{3/2}$.

The linearized equations of continuity do not change.

Finally there is no change in the equation relating s_1 and s_2 to E, unless $\omega/k = c$. To see this we note that the vector potential is parallel to Ox. Let A be its magnitude. Then

$$\frac{\partial^2 A}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{1}{c}\{4\pi N\epsilon(1+s_1)(U+u_1)+4\pi N\epsilon(1+s_2)(-U+u_2)\}. \quad (17)$$

For the scalar potential ϕ,

$$\frac{\partial^2 \phi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = -4\pi N\epsilon(s_1+s_2). \quad (18)$$

We operate on (17) with $-\partial/c\,\partial t$ and on (18) with $-\partial/\partial x$. Then since,

$$E = -\frac{1}{c}\frac{\partial A}{\partial t}-\frac{\partial \phi}{\partial x}$$

we find, on adding and linearizing, that

$$\frac{\partial^2 E}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 4\pi N\epsilon\left[\frac{1}{c^2}\frac{\partial}{\partial t}(u_1+Us_1)+\frac{\partial s_1}{\partial x}+\frac{1}{c^2}\frac{\partial}{\partial t}(u_2-Us_2)+\frac{\partial s_2}{\partial x}\right]. \quad (19)$$

But, according to the linearized equations of continuity (6) and (7),

$$\frac{\partial}{\partial x}(u_1+Us_1) = -\frac{\partial s_1}{\partial t}, \qquad \frac{\partial}{\partial x}(u_2-Us_2) = -\frac{\partial s_2}{\partial t},$$

and it follows that

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = 4\pi N\epsilon\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\int^x (s_1+s_2)\,dx. \quad (20)$$

Equation (20) is equivalent to equation (3), unless the wave motion is such that

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = 0$$

and

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)(s_1+s_2) = 0,$$

that is, unless the disturbance propagates with the speed of light.

Since the only change made in the relativistic treatment is to replace m_0 by $m_0(1-U^2/c^2)^{-3/2}$ we find that the critical wavelength now becomes

$$\lambda_{\min} = \sqrt{\left(\frac{\pi m_0 U^2}{2N\epsilon^2}\right)}\left(1-\frac{U^2}{c^2}\right)^{-3/4}. \quad (21)$$

4. The amplification of oscillations

We now return to the non-relativistic case. Suppose that two uniform, ionized, semi-infinite streams collide with one another. Once again let there be N protons and N electrons per unit volume, and let $+U$ and $-U$ be the undisturbed velocities in the two streams. Let the first impact occur at $x = 0$ at time $t = 0$.

We assume further that the temperature in each stream is initially zero, but that there are present in it random electrostatic oscillations, whose energy per unit mass is small compared with $\frac{1}{2}U^2$. We shall try to find out how long it is likely to be before these oscillations are amplified sufficiently to make the linear approximation break down. When this happens the electrostatic energy will be comparable with the energy of counterstreaming, and the latter may then be expected to be brought to a halt. The calculation should give a closer estimate of the possible distance of interpenetration of the electrons in the two streams.

The neglect of the initial temperature T_0 is justified provided $U^2 \gg kT_0/m_0$. For, as Bohm and Gross have shown (see, for instance, Spitzer (1956)), the thermal motions do not appreciably affect plasma oscillations whose wavelengths are much in excess of $\sqrt{(kT_0/N\epsilon^2)}$, whereas the oscillations that are amplified most readily in the present case are those with wavelengths of the order of $\sqrt{(m_0 U^2/N\epsilon^2)}$.

The introduction of dimensionless variables will simplify the further working. We set

$$\tau = \Omega t, \qquad X = \Omega x/U, \qquad \mathscr{E} = \epsilon E/m_0 \Omega U, \qquad v_1 = u_1/U, \qquad v_2 = u_2/U.$$

(The meanings of E, Ω, ϵ, m_0, u_1 and u_2 are the same as in § 2.) The equations of § 2 become

$$\left.\begin{aligned} \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial X}\right)v_1 &= \mathscr{E}, & \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X}\right)v_2 &= \mathscr{E}, \\ \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial X}\right)s_1 + \frac{\partial v_1}{\partial X} &= 0, & \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X}\right)s_2 + \frac{\partial v_2}{\partial X} &= 0, \\ \frac{\partial \mathscr{E}}{\partial X} &= s_1 + s_2, \end{aligned}\right\} \tag{22}$$

where s_1 and s_2 also have the same meaning as in § 2. The equations (22) combine to give

$$\left.\begin{aligned} \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial X}\right)^2 s_1 + s_1 &= -s_2, \\ \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X}\right)^2 s_2 + s_2 &= -s_1, \end{aligned}\right\} \tag{23}$$

or

$$\left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial X}\right)^2 \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X}\right)^2 s_{1,2} + \left(\frac{\partial}{\partial \tau} + \frac{\partial}{\partial X}\right)^2 s_{1,2} + \left(\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X}\right)^2 s_{1,2} = 0 \tag{24}$$

(the symbol $s_{1,2}$ stands for either s_1 or s_2).

A set of coordinates alternative to X and τ is given by

$$\xi = \tau - X, \qquad \eta = \tau + X,$$

and leads to

$$\frac{\partial}{\partial \tau} - \frac{\partial}{\partial X} = 2\frac{\partial}{\partial \xi}, \qquad \frac{\partial}{\partial \tau} + \frac{\partial}{\partial X} = 2\frac{\partial}{\partial \eta}.$$

Equations (23) then become

$$4\frac{\partial^2 s_1}{\partial \eta^2} + s_1 = -s_2, \tag{25}$$

$$4\frac{\partial^2 s_2}{\partial \xi^2} + s_2 = -s_1, \tag{26}$$

and (24) becomes

$$4\frac{\partial^4 s_{1,2}}{\partial \xi^2 \partial \eta^2} + \frac{\partial^2 s_{1,2}}{\partial \xi^2} + \frac{\partial^2 s_{1,2}}{\partial \eta^2} = 0. \tag{27}$$

Now the first stream is moving in the direction of increasing X; at time τ its front surface is at $X = \tau$, while the front surface of the second stream is at $X = -\tau$. Counterstreaming therefore occurs only in the part of the half-plane $\tau > 0$ which lies between the lines $X = \tau$ and $X = -\tau$, or, alternatively, between $\xi = 0$ and $\eta = 0$ (region A in figure 1).

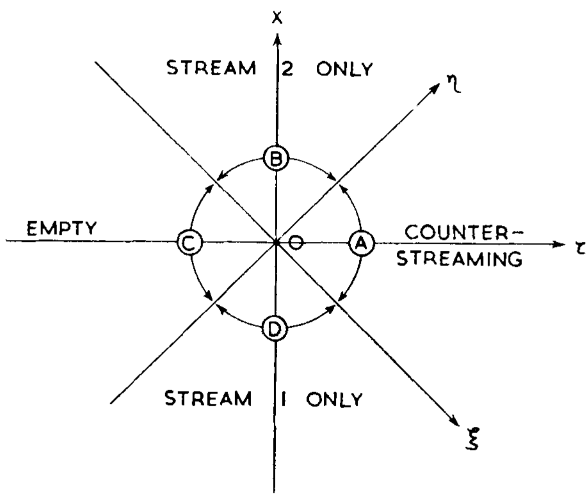

Figure 1. The nature of the motion in various regions.

Regions B and D in figure 1 are occupied only by streams 2 and 1, respectively. Region C is empty. Our equations therefore apply only in region A; to find the appropriate solutions adequate boundary conditions must be known on the positive parts of $O\xi$ and $O\eta$. These boundary conditions are determined by the oscillations present in the single streams occupying regions B and D.

In figure 2 the diagram is re-drawn with the ξ-axis horizontal and the η-axis vertical. Suppose we seek the solution of the equations within a region to the right of and above a curve PQ whose gradient never takes a finite positive value. The solution of equation (25) may be written

$$s_1(\xi,\eta) = s_1(\xi,\eta_0)\cos\tfrac{1}{2}\eta + 2\frac{\partial s_1}{\partial\eta}(\xi,\eta_0)\sin\tfrac{1}{2}\eta - 2\int_{\eta_0}^{\eta} s_2(\xi,u)\sin\tfrac{1}{2}(\eta-u)\,du, \quad (28)$$

and this shows that the value of s_1 at a typical point R is determined solely by the values of s_1 and $\partial s_1/\partial\eta$ at the point S_0, directly below R on PQ, and by the value of s_2 at all points of RS_0.

Repetition of the argument shows that the value of s_2 at R', say, depends only on the values of s_2 and $\partial s_2/\partial\xi$ at T_0', which lies on PQ to the left of R', and on all the values taken by s_1 on $T_0'R'$. Continuing in this way we find that the values of s_1 and s_2 at R are determined completely by the values of s_1 and $\partial s_1/\partial\eta$, s_2 and $\partial s_2/\partial\xi$ on the stretch $T_0 S_0$ of PQ, where T_0 lies to the left of R and on PQ. In fact these values define the function within the area $RS_0 T_0$. It follows that the solution within the region $0 \leqslant \xi \leqslant \xi_0$, $0 \leqslant \eta \leqslant \eta_0$ is fully determined when s_1 and $\partial s_1/\partial\eta$ are known on the stretch of the ξ-axis from 0 to ξ_0, and when s_2 and $\partial s_2/\partial\xi$ are known on the η-axis, from 0 to η_0.

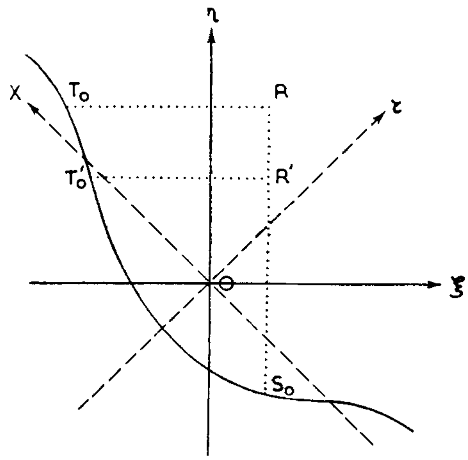

Figure 2. The domain of dependence for a point R.

But the η-axis has the equation $X = \tau$ in the other set of coordinates. Its positive part therefore represents the front surface of stream 1 from time $\tau = 0$ onwards. The required boundary conditions are therefore equivalent to a knowledge of s_2 and $\partial s_2/\partial\xi$ at this front surface, and of s_1 and $\partial s_1/\partial\eta$ at the other front surface, that is, to a knowledge of the conditions at various times in the part of stream 2 which is just about to collide with stream 1, and vice versa.

The equations are linear and so a general solution may be built up by the addition of various basic solutions corresponding to simple boundary conditions. Finally the solution for the first quadrant of the (ξ,η)-plane due to two semi-infinite streams will be the same as that due to two infinite streams of the same density provided only the same boundary values on the positive halves of the ξ- and η-axes apply in the two cases.

We shall now find the solution that corresponds to the following simple boundary conditions:

$$s_1 = \delta(\xi - \xi_0), \qquad \partial s_1/\partial\eta = 0,$$

on the positive half of the ξ-axis;

$$s_2 = 0, \qquad \partial s_2/\partial\xi = 0$$

on the positive half of the η-axis. Here $\delta(\xi)$ stands for the Dirac delta function.

Let P be the point $(\xi_0, 0)$. The solutions we shall find will be valid only to the right of and above P (see figure 3). In this region, the solution with the above boundary conditions is the same as that given by the boundary conditions $s_1 = \delta(\xi - \xi_0)$, $\partial s_1/\partial\eta = 0$, $s_2 = 0$, $\partial s_2/\partial\xi = 0$, *on the line AB* through P, provided the slope of this line is negative, say -1. This alternative set of conditions leads to the same values on $P\xi$ and $P\eta'$, since the values of s_2 and $\partial s_2/\partial\xi$ are zero at all points to the right of AP and to the left of $P\eta'$, and therefore by continuity on $P\eta'$. Similarly the values of s_1 and $\partial s_1/\partial\eta$ are zero at all points above PB and below $P\xi$, with the exception of the point P.

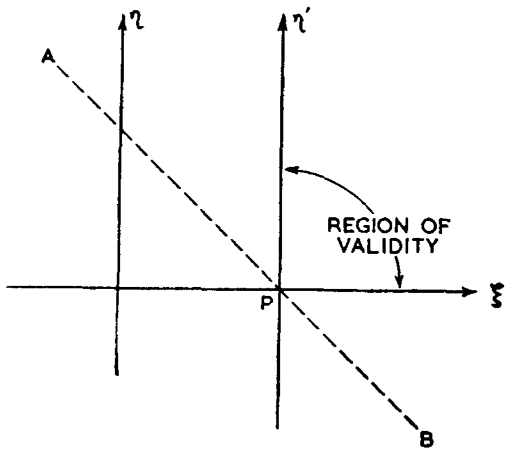

Figure 3. Region of validity of the basic solution.

The nature of the solution corresponding to these boundary conditions is not affected by the exact value of ξ_0; the solutions for the points ξ_0 and ξ_0' are identical except for a relative displacement $\xi_0 - \xi_0'$ parallel to $O\xi$. We therefore choose $\xi_0 = 0$ for the sake of simplicity. The line AB then has the equation $\xi + \eta = 0$, or $\tau = 0$ in (X, τ) coordinates.

The equivalent boundary conditions are now

$$\left.\begin{aligned} s_1 &= \delta(\tau - X), & \frac{\partial s_1}{\partial\eta} &= \frac{1}{2}\left(\frac{\partial s_1}{\partial\tau} + \frac{\partial s_1}{\partial X}\right) = 0, \\ s_2 &= 0, & \frac{\partial s_2}{\partial\xi} &= \frac{1}{2}\left(\frac{\partial s_2}{\partial\tau} - \frac{\partial s_2}{\partial X}\right) = 0, \end{aligned}\right\} \text{on } \tau = 0, \qquad (29)$$

or

$$\left.\begin{array}{ll} s_1 = \delta(X), & \dfrac{\partial s_1}{\partial \tau} = -\dfrac{\partial s_1}{\partial X} \\ s_2 = 0, & \dfrac{\partial s_2}{\partial \tau} = 0, \end{array}\right\} \text{ on } \tau = 0. \tag{29}$$

With the aid of the first of the equations (23) the third and fourth conditions (29) may be written

$$\left.\begin{array}{l} \left(\dfrac{\partial}{\partial \tau} + \dfrac{\partial}{\partial X}\right)^2 s_1 = -s_1 = -\delta(X), \\ \dfrac{\partial}{\partial \tau}\left(\dfrac{\partial}{\partial \tau} + \dfrac{\partial}{\partial X}\right)^2 s_1 = -\dfrac{\partial s_1}{\partial \tau} = \dfrac{\partial s_1}{\partial X}, \end{array}\right\} \text{ on } \tau = 0. \tag{29a}$$

A possible solution of the differential equation (27) for s_1 is

$$s_1 = e^{i[KX + \mu(K)\tau]}$$

provided $\mu(K)$ satisfies

$$(\mu^2 - K^2)^2 - 2(\mu^2 + K^2) = 0,$$

that is, $$\mu^4 - 2\mu^2(K^2+1) + (K^4 - 2K^2) = 0. \tag{30}$$

(This equation is the dimensionless equivalent of (14).) Corresponding to any given positive value of K there are four values of μ, of which two may be imaginary. The most general solution of (27) is, in the complex notation,

$$s_1 = \sum_{r=1}^{4} \int_{K=0}^{\infty} \alpha_r(K) e^{i[KX + \mu_r(K)\tau]}\, dK. \tag{31}$$

The four α_r coefficients for each K have now to be determined from the boundary conditions for $\tau = 0$. In the complex notation,

$$\delta(X) = \frac{1}{\pi}\int_0^\infty e^{iKX}\, dK = s_1(X, 0)$$

so that $$\frac{\partial}{\partial X} s_1(X,0) = \frac{1}{\pi}\int_0^\infty K e^{iKX}\, dK.$$

Hence, from (29) and (31),

$$\sum_{r=1}^{4} \int_0^\infty \alpha_r(K) e^{iKX}\, dK = \frac{1}{\pi}\int_0^\infty e^{iKX}\, dK.$$

Comparison of coefficients then shows that

$$\sum_{r=1}^{4} \alpha_r(K) = \frac{1}{\pi}.$$

Setting $$\beta_r = \pi\alpha_r, \tag{32}$$

we have $$\sum_{r=1}^{4} \beta_r = 1. \tag{33}$$

The other conditions give that

$$\sum_{r=1}^{4} \beta_r \mu_r = -K, \tag{34}$$

$$\sum_{r=1}^{4} \beta_r[K+\mu_r]^2 = 1, \tag{35}$$

$$\sum_{r=1}^{4} \mu_r \beta_r[K+\mu_r]^2 = -K. \tag{36}$$

Now the operation $(35)-2K\times(34)-K^2\times(33)$ gives

$$\sum_{r=1}^{4} \beta_r \mu_r^2 = K^2+1, \tag{35 a}$$

and the operation $(36)-2K\times(35\,\text{a})-K^2\times(34)$ gives

$$\sum_{r=1}^{4} \beta_r \mu_r^3 = -K^3-3K. \tag{36 a}$$

But the roots of (30) occur in pairs, such that

$$\mu_1 = -\mu_2, \qquad \mu_3 = -\mu_4, \qquad \mu_1^2+\mu_3^2 = 2(K^2+1).$$

After some manipulation the solution for β_1 is found to be

$$\beta_1 = \frac{1}{4} - \frac{K}{4\mu_1}\left\{\frac{\mu_1^2-K^2+1}{\mu_1^2-K^2-1}\right\}. \tag{37}$$

We are especially interested in the disturbances that are amplified most rapidly, that is, in those for which μ_1 takes its numerically largest imaginary value, which is readily seen to be $\pm\frac{1}{2}i$, and occurs when $K = \frac{1}{2}\sqrt{3}$. In this case $\beta_1 = \frac{1}{4}$. It is seen that $\mu_1 = -\frac{1}{2}i$ is the appropriate coefficient for an amplified wave.

In general, when μ_1 is imaginary, the second term on the right-hand side of (37) gives the imaginary part of β_1. This is of the first order in $\kappa \equiv K-\frac{1}{2}\sqrt{3}$, and so $|\beta| = \frac{1}{4}$, to the second order in κ

The solution of (30) in terms of κ is

$$\begin{aligned}\mu_1^2 &= (\tfrac{1}{2}\sqrt{3}+\kappa)^2+1-\sqrt{\{4(\tfrac{1}{2}\sqrt{3}+\kappa)^2+1\}} \\ &= -\tfrac{1}{4}(1-3\kappa^2+\dots)\end{aligned}$$

to the second order in κ, and so

$$\mu_1 = -\tfrac{1}{2}i(1-\tfrac{3}{2}\kappa^2+\dots) \tag{38}$$

also to the second order.

Finally, the imaginary part of β_1 is given by

$$\begin{aligned}\mathscr{I}(\beta_1) &= \frac{\frac{1}{2}\sqrt{3}+\kappa}{4(-\frac{1}{2})}\left\{\frac{-\frac{1}{4}-(\frac{1}{2}\sqrt{3}+\kappa)^2+1}{-\frac{1}{4}-(\frac{1}{2}\sqrt{3}+\kappa)^2-1}\right\} \\ &= -\tfrac{3}{8}\kappa\end{aligned}$$

to the first order in κ, so that

$$\beta_1 = \tfrac{1}{4}-\tfrac{3}{8}i\kappa$$

and

$$\alpha_1 = \frac{1}{4\pi}\left(1-\frac{3i\kappa}{2}\right). \tag{39}$$

For large values of τ the solution therefore asymptotically tends to

$$
\begin{aligned}
s_1 &= \int_K \alpha_1(K) e^{i(KX+\mu_1\tau)}\, dK \\
&\sim \frac{1}{4\pi} e^{i\sqrt{3}X/2} \int_\kappa \left(1 - \frac{3i\kappa}{2}\right) e^{\frac{1}{2}(1-3\kappa^2/2)\tau} e^{i\kappa X}\, d\kappa \\
&\sim \frac{1}{4\pi} e^{i\sqrt{3}X/2} e^{\tau/2} \int_\kappa e^{-3\kappa^2\tau/4}\, e^{i\kappa X} d\kappa \\
&\sim \frac{1}{\sqrt{(12\pi\tau)}} e^{i\sqrt{3}X/2}\, e^{\tau/2} e^{-X^2/3\tau}.
\end{aligned} \tag{40}
$$

The solution is valid only at points where $|X| < \tau$, that is at all points where there is double streaming.

If the condensation $s_1(X,\tau)$ is produced by an input function $\delta(\xi - 2u)$ on the line $\eta = 0$, the corresponding point P in figure 3 has the coordinates $\xi = 2u$, $\eta = 0$, or $X = -u$, $\tau = u$. The expression for the condensation is now

$$
s_1(X,\tau) = \frac{1}{\sqrt{(12\pi\tau)}} \exp\left[\frac{i\sqrt{3}}{2}(X+u) + \frac{\tau - u}{2} - \frac{(X+u)^2}{3\tau}\right]. \tag{41}
$$

In general, let the input function be $e^{iq\tau_b}$ at the point $X = -\tau_b$ at time $\tau = \tau_b$, for $\tau_b > 0$; this instant is on the line $X + \tau \equiv \eta = 0$. The corresponding condensation will be

$$
s_1(X,\tau) = \int_{\tau_b = 0}^{\infty} \frac{e^{iq\tau_b}}{\sqrt{[12\pi(\tau - \tau_b)]}} \exp\left[\frac{i\sqrt{3}}{2}(X+\tau_b) + \frac{\tau - \tau_b}{2} - \frac{(X+\tau_b)^2}{3(\tau - \tau_b)}\right] d\tau_b. \tag{42}
$$

The integrand in (42) has the form $R(\tau_b)\exp[P(\tau_b) + iQ(\tau_b)]$, where P, Q and R are real, and $P(\tau_b)$ is a decreasing function of τ_b. The integral may therefore be approximated by

$$
R(0)\exp[P(0) + iQ(0)] \int_0^{\infty} \exp[P'(0) + iQ'(0)]\tau_b\, d\tau_b
$$

and so

$$
s_1(X,\tau) \doteqdot \frac{1}{\sqrt{(12\pi\tau)}} \frac{\exp\left\{i\dfrac{\sqrt{3}X}{2}\right\} \exp\left\{\tau\left(\dfrac{1}{2} - \dfrac{X^2}{3\tau^2}\right)\right\}}{i\left(\dfrac{\sqrt{3}}{2} + q\right) - \dfrac{1}{2} - \left(\dfrac{2X}{3\tau} + \dfrac{X^2}{3\tau^2}\right)}.
$$

This is large only when $\tau \gg 1$, and in that case the greatest values of s_1 occur where $|X| \ll \tau$, giving, at such points,

$$
s_1(X,\tau) \doteqdot \frac{1}{\sqrt{(12\pi\tau)}} \frac{\exp\left\{i\dfrac{\sqrt{3}X}{2}\right\} \exp\left\{\tau\left(\dfrac{1}{2} - \dfrac{X^2}{3\tau^2}\right)\right\}}{i\left(\frac{1}{2}\sqrt{3} + q\right) - \frac{1}{2}}. \tag{43}
$$

When the input function is given by

$$
f(\tau_b) = \int_{-\infty}^{\infty} g(q) e^{iq\tau_b}\, dq \tag{44}
$$

we find

$$s_1(X,\tau) \doteqdot -\frac{\exp\left\{i\frac{\sqrt{3}X}{2}+\frac{\tau}{2}-\frac{X^2}{3\tau}\right\}}{\sqrt{(12\pi\tau)}}\int_{-\infty}^{\infty}\frac{g(q)\,dq}{\frac{1}{2}-i(q+\frac{1}{2}\sqrt{3})}. \tag{45}$$

As we shall show, the plasma oscillation in stream 1 before the collision generates an input function which may be described in the form of a Fourie integral, as in (44). The function $g(q)$ is such that

$$\langle g(q)g^*(q')\rangle = [w(q)w(q')]^{1/2}\,\delta(q-q') \tag{46}$$

where $\delta(q)$ is once again the Dirac delta function, and the diamond-shaped brackets $\langle\ldots\rangle$ stand for "expectation value of...". There is no phase relation between any $g(q)$ and any $g(q')$, unless $q = q'$. It follows that

$$\langle s_1 s_1^*(X,\tau)\rangle = \frac{\exp(\tau-2X^2/3\tau)}{12\pi\tau}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\langle g(q)g^*(q')\rangle\,dq\,dq'}{[\frac{1}{2}-i(q+\frac{1}{2}\sqrt{3})][\frac{1}{2}+i(q'+\frac{1}{2}\sqrt{3})]}$$

$$= \frac{\exp(\tau-2X^2/3\tau)}{12\pi\tau}\int_{-\infty}^{\infty}\frac{w(q)\,dq}{1+q\sqrt{3}+q^2}. \tag{47}$$

It only remains to express this result in terms of the internal motion of stream 1 before the collision. This is best described by means of a position coordinate ξ at rest in the stream, and the time coordinate τ. The linearized, dimensionless equations of the oscillation are

$$\partial v/\partial\tau = \mathscr{E} \quad \text{(motion)}, \tag{48 a}$$

$$\partial s/\partial\tau + \partial v/\partial\xi = 0 \quad \text{(continuity)}, \tag{48 b}$$

$$\partial\mathscr{E}/\partial\xi = s \quad \text{(Poisson)}. \tag{48 c}$$

Hence

$$\partial^2\mathscr{E}/\partial\tau^2 + \mathscr{E} = 0,$$

and

$$\mathscr{E} = G(\xi)e^{-i\tau} \tag{49}$$

in the complex notation. The form of the solution shows the well-known fact that the oscillations in the single stream have a frequency independent of the wave number, and so have zero group velocity. They cannot therefore propagate energy, and the part of stream 1 that has not yet collided remains unaffected by the part that has done so.

From (48 a) and (49) it follows that

$$v = iG(\xi)e^{-i\tau}, \tag{50}$$

and from (48 c) and (49) that

$$s = G'(\xi)e^{-i\tau}. \tag{51}$$

The average electrostatic energy per unit volume is

$$\frac{1}{16\pi}EE^* = \frac{1}{16\pi}\frac{m_0^2U^2}{\epsilon^2}\mathscr{E}\mathscr{E}^* = \tfrac{1}{4}Nm_0\,U^2GG^*(\xi),$$

and the average kinetic energy per unit volume of the electron gas relative to the stream as a whole is

$$\tfrac{1}{4}Nm_0\,uu^* = \tfrac{1}{4}Nm_0\,U^2vv^* = \tfrac{1}{4}Nm_0\,u^2GG^*(\xi).$$

The energy per unit volume of the electron plasma is therefore

$$\tfrac{1}{2}Nm_0\,U^2GG^*(\xi).$$

The input function for the motion in the region of counterstreaming is now $G'(\xi)e^{-i\tau}$, evaluated on the line $X(\equiv \tau-\xi) = -\tau$, or $\xi = 2\tau$, and is equal to $G'(2\tau)e^{-i\tau} \equiv f(\tau)$.

But the field in the incident stream may be represented by

$$G(\xi)e^{-i\tau} = e^{-i\tau}\int_{-\infty}^{\infty} \Gamma(l)\, e^{il\xi}\, dl; \tag{52}$$

if it is a random field there are no phase relations between any $\Gamma(l)$ and $\Gamma(l')$, unless $l = l'$. We have once again

$$\langle \Gamma(l)\Gamma^*(l')\rangle = [W(l)W(l')]^{1/2}\,\delta(l-l'), \tag{53}$$

and

$$\begin{aligned}\langle GG^*(\xi)\rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \Gamma(l)\Gamma^*(l')\rangle\, dl\,dl' \\ &= \int_{-\infty}^{\infty} W(l)\,dl.\end{aligned} \tag{54}$$

The average internal energy per unit volume present in the dimensionless wave number range $(l, l+dl)$ is $\frac{1}{2}Nm_0\, U^2 W(l)\,dl$; the total internal energy density of the plasma is

$$T_{\text{pl}} = \tfrac{1}{2}Nm_0\, U^2 \int_{-\infty}^{\infty} W(l)\,dl. \tag{55}$$

Further

$$G'(\xi)\,e^{-i\tau} = ie^{-i\tau}\int_{-\infty}^{\infty} l\Gamma(l)e^{il\xi}\,dl$$

and so

$$\begin{aligned}f(\tau) &= ie^{-i\tau}\int_{-\infty}^{\infty} l\Gamma(l)e^{2il\tau}\,dl \\ &\equiv i\int_{-\infty}^{\infty} (\tfrac{1}{4}q+\tfrac{1}{4})\Gamma(\tfrac{1}{2}q+\tfrac{1}{2})\, e^{iq\tau}\,dq.\end{aligned} \tag{56}$$

We can now identify $g(q)$ with $i(\frac{1}{4}q+\frac{1}{4})\Gamma(\frac{1}{2}q+\frac{1}{2})$. Since there are no phase relations among the Γ's, there are none among the g's either, as required. It follows from (47) that

$$\begin{aligned}\langle s_1 s_1^*(X,\tau)\rangle &= \frac{\exp(\tau-2X^2/3\tau)}{12\pi\tau}\,\frac{1}{16}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{(q+1)^2}{1+q\sqrt{3}+q^2} \\ &\qquad\times \langle \Gamma(\tfrac{1}{2}q+\tfrac{1}{2})\Gamma^*(\tfrac{1}{2}q'+\tfrac{1}{2})\rangle\, dq\,dq' \\ &= \frac{\exp(\tau-2X^2/3\tau)}{192\pi\tau}\int_{-\infty}^{\infty} \frac{(q+1)^2}{1+q\sqrt{3}+q^2}\, 2W(\tfrac{1}{2}q+\tfrac{1}{2})\,dq \\ &= \frac{\exp(\tau-2X^2/3\tau)}{48\pi\tau}\left\{\int_{-\infty}^{\infty} W(l)\,dl + O(1/\bar{l})\right\}\end{aligned} \tag{57}$$

where $\bar{l}$ is the effective dimensionless wave number range of the internal oscillations. Now $\frac{1}{2}Nm_0\, U^2 = T_{\text{kin}}$ is the initial kinetic energy of motion of the electrons in stream 1 relative to the (x, t) system of coordinates (which is symmetrical with respect to the two streams). It follows from (55) and (57) that

$$\langle s_1 s_1^*(X,\tau)\rangle = \frac{\exp(\tau-2X^2/3\tau)}{48\pi\tau}\,\frac{T_{\text{pl}}}{T_{\text{kin}}}. \tag{58}$$

It was one of the initial assumptions in setting up the linearized equations of motion that the condensation s_1 is much less than unity. Equation (58) shows that this is most probably no longer true when

$$\frac{\exp(\tau - 2X^2/3\tau)}{48\pi\tau} \geqslant \frac{T_{\text{kin}}}{T_{\text{pl}}}. \tag{59}$$

However, when s_1 is of order unity, the energy per unit mass of the plasma oscillations in the double stream will be comparable with the initial relative kinetic energy per unit mass of the two streams; since the former must grow at the expense of the latter it follows that the relative motion will by that time have been stopped to a large extent. The minimum value τ_{crit} which satisfies (59) therefore leads to an estimate of the length of time for which the two streams can interpenetrate, and of the depth to which they can do so.

The smallest value of τ for which (59) holds occurs at $X = 0$, that is at the position where the two streams made first contact, and here

$$\frac{\exp(\tau_{\text{crit}})}{48\pi\tau_{\text{crit}}} = \frac{T_{\text{kin}}}{T_{\text{pl}}},$$

or

$$\tau_{\text{crit}} \doteqdot \frac{a}{a-1}(a + \log a - 1), \tag{60}$$

where

$$a = \log\{48\pi(T_{\text{kin}}/T_{\text{pl}})\}. \tag{61}$$

To take some examples, when $T_{\text{pl}}/T_{\text{kin}} = 10^{-1}$, 10^{-3} or 10^{-5}, $\tau_{\text{crit}} = 9{\cdot}6$, $14{\cdot}6$ or $19{\cdot}6$, respectively. The times corresponding to these values are given by

$$t_{\text{crit}} = \frac{\tau_{\text{crit}}}{\Omega} = \tau_{\text{crit}}\sqrt{\left(\frac{m_0}{4\pi N\epsilon^2}\right)};$$

since $2U$ is the initial relative velocity of the streams the corresponding distances of penetration of stream 1 into stream 2 are given by

$$2Ut_{\text{crit}} = 2\tau_{\text{crit}}\sqrt{\left(\frac{m_0 U^2}{4\pi N\epsilon^2}\right)}. \tag{62}$$

Any plasma oscillation initially present in stream 2 will lead to a further contribution to the right-hand side of (58). If the oscillations in the two streams have roughly equal energies, the effect will be to multiply that side by 2, and this leads to a decrease by $\log_e 2 = 0{\cdot}69$ in the estimate for τ_{crit}.

The more accurate estimate for the distance of interpenetration of the streams, given in (62), only differs by the numerical factor $\sqrt{(2/\pi)}\tau_{\text{crit}}$, from the earlier, cruder estimate given in (15). The numerical examples that have been quoted show that in any particular case τ_{crit} depends, but only rather insensitively, on the ratio of the energies of the plasma oscillations present in the streams before collision to the energy of their relative motion.

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